§7 Произв-я по напр-ию и градиент

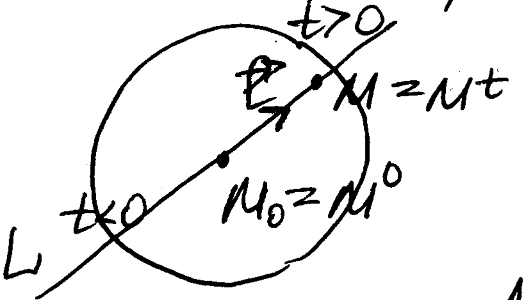
6.1

Пусть $u=f(M)=f(x,y)$: $D_f \supset$ окр т. M_0

Зададим напр-ие с помощью

$\vec{l}=\{l_1,l_2\}$: $|\vec{l}|=\sqrt{l_1^2+l_2^2}=1$

$$\Rightarrow \underbrace{l_1=\cos\alpha,\ l_2=\cos\beta}_{\text{(напр-ие cos-ы)}},\ \alpha,\beta\in[0,\pi]$$



т. M_0, $\vec{l}$ опр-т прямую L

Пусть $[M_0,M]\subset L$ — напр-ый отр

величина $[M_0,M]$ → $$M_0M \equiv \begin{cases} +|\vec{M_0M}|, & \vec{M_0M}\uparrow\uparrow\vec{l} \\ -|\quad|, & \uparrow\downarrow \end{cases}$$

Опр Если сущ $\lim\limits_{\substack{M\to M_0\\ M\in L}}\dfrac{f(M)-f(M_0)}{M_0M}$, то он наз

пр-ой ф-ии $u=f(M)$ в т. M_0 по напр-ю $\vec{l}$

Обозн $\dfrac{\partial u}{\partial l}(M_0)$, $u_{\vec{l}}(M_0)$

Утв Если $u(M)$ дифф в т. M_0, то $\forall \vec{l}\ \exists$ ~~$\{\cos\alpha,\cos\beta\}$~~

$$\frac{\partial u}{\partial l}(M_0)=\frac{\partial u}{\partial x}(M_0)\cos\alpha+\frac{\partial u}{\partial y}(M_0)\cos\beta$$

$(\exists\ u=)$

6.2

Δ Пусть
$M_0 = M_0(x_0, y_0) \Rightarrow L: \begin{cases} x = \overbrace{x_0 + t\cos\alpha}^{x(t)} \\ y = \overbrace{y_0 + t\cos\beta}^{y(t)} \end{cases} \quad t \in \mathbb{R}$

$M^t \equiv M(x(t), y(t))$
$\Rightarrow M_0(x_0, y_0) = M(x(0), y(0)) = M^0$

На прямой L
$u = f(M^t) = f(x(t), y(t)) \equiv \varphi(t)$

За~~метим~~, что $|M^0M^t| = \sqrt{[x(t) - x_0]^2 + [\ \ ]^2} =$

$= \sqrt{\underbrace{t^2\cos^2\alpha + t^2\cos^2\beta}_{=1}} = \sqrt{t^2} = |t|$

а значит
$M^t \to M^0 \Leftrightarrow |M^0M^t| \to 0 \Leftrightarrow t \to 0$

Кроме того, т.к. $\operatorname{sign} M^0M^t = \operatorname{sign} t$
то $M^0M^t = t$

Тогда
$\frac{\partial u}{\partial l}(M_0) = \lim\limits_{M^t \to M^0} \frac{f(M^t) - f(M^0)}{M^0M^t} = \lim\limits_{t \to 0} \frac{\varphi(t) - \varphi(0)}{t - 0} =$

$= \underbrace{\frac{d\varphi}{dt}(0)}_{} = f_x(\overset{x_0}{x(0)}, \overset{y_0}{y(0)}) \cdot \overset{\cos\alpha}{x_t'(0)} + \text{———} \text{ и } \text{———} =$

($\exists$ по теор о дифф-ти сл-й ф-ии)

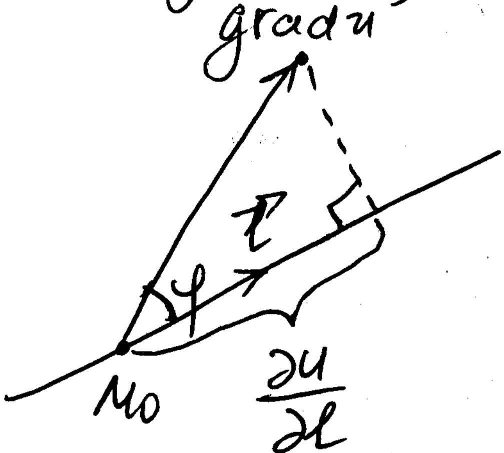
$= u_x(M_0)\cos\alpha + u_y(M_0)\cos\beta$ ~~4~~ 6.3

Опр Градиентом ф-ции $u(M)$ в т. M_0 наз вектор

$$\{u_x(M_0), u_y(M_0)\} \equiv \operatorname{grad} u\big|_{M_0}$$

С помощью grad-та получаем

$$\frac{\partial u}{\partial l} = (\operatorname{grad} u, \vec{l}) = \underbrace{|\operatorname{grad} u| \cdot \overset{=1}{\cancel{|\vec{l}|}} \cdot \overset{|\,|\le 1}{\cos\varphi}} = \text{Пр}_{\vec{l}} \operatorname{grad} u$$



$$\left|\frac{\partial u}{\partial l}\right| \le |\operatorname{grad} u|$$

причём $\left(\frac{\partial u}{\partial l}\right)_{max} = +|\operatorname{grad} u|$ при $\varphi = 0$ (дост-ся)

$(\quad)_{min} = -|\quad|$ при $\varphi = \pi$

(1) $u = x^2 + y^2$, $M_0(1,2)$

$u_x(M_0) = 2$, $u_y(M_0) = 4 \Rightarrow \operatorname{grad} u\big|_{M_0} = \{2, 4\}$

Опр $\{M \in D_u \mid u(M) = C\}$ — линия уровня ф-ии u

Найдём л.у., пр-ую через т. M_0

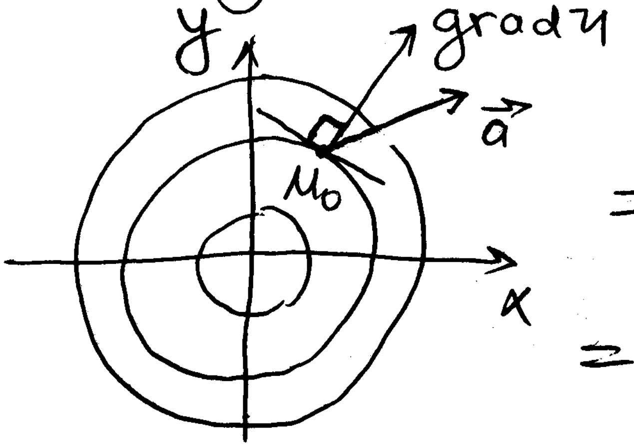
6.4

$u(M)=C = u(M_0)=5$

$x^2+y^2=5$ — окр $R=\sqrt{5}$

$x x_0 + y y_0 = 5$ — ур-ие кас-й

$x+2y=5 \Rightarrow \vec{n}=\{1,2\} \parallel \operatorname{grad} u\big|_{M_0}$



Пусть $\vec{a}=\{1,1\}$, $\vec{l}=\frac{\vec{a}}{|\vec{a}|}$

$$\Rightarrow \frac{\partial u}{\partial a}(M_0) \equiv \frac{\partial u}{\partial l}(M_0) = \underbrace{u_x(M_0)}_{=2}\frac{1}{\sqrt{2}} + \underbrace{u_y(M_0)}_{=4}\frac{1}{\sqrt{2}} = 3\sqrt{2}$$

§8 Частные пр-ые и диф-лы высших порядков

Пусть $u=f(M)=f(x_1,x_2)$: $\frac{\partial u}{\partial x_i}$ опр в окр т. M_0

$$\Rightarrow \frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial x_i}(M) = \frac{\partial u}{\partial x_i}(x_1,x_2)$$

Опр $\frac{\partial^2 u}{\partial x_j \partial x_i}(M_0) \equiv \frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial x_i}(M)\right)\Big|_{M_0}$

— ~~вторая~~ 2-я ЧП ф-ции u в т. M_0

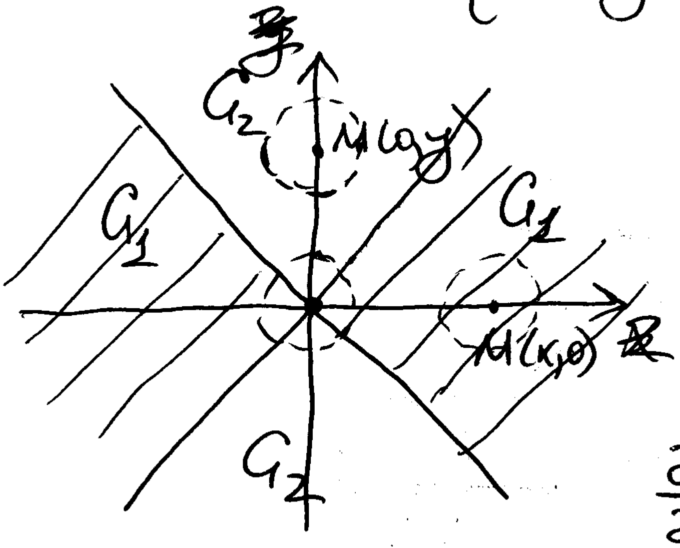
$i \neq j \Rightarrow \frac{\partial^2 u}{\partial x_j \partial x_i} \equiv u_{x_i x_j}$ — смеш-я ЧП

$i = j \Rightarrow \frac{\partial^2 u}{\partial x_i^2} \equiv u_{x_i^2}$

(Ан-но опр-ся ~~пр-ые~~ ЧП старших порядков)

6.5

1. $u(x,y)=\begin{cases} +xy, & |y|\le|x| \leftarrow G_1 \\ -xy & |y|>|x| \leftarrow G_2 \end{cases}$



$$u_{xy}(0,0)=\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}(x,y)\right]_{\substack{x=0\\y=0}}\equiv$$

$$\equiv\frac{d}{dy}\left[\frac{\partial u}{\partial x}(0,y)\right]_{y=0}$$

$$\frac{\partial u}{\partial x}(0,y)\equiv\frac{d}{dx}[u(x,\overset{const}{y})]_{x=0}$$

Пусть $y\neq 0 \Rightarrow M(0,y)\in G_2$, причём $\exists\,\varepsilon>0$:

$O_\varepsilon(M)\subset G_2 \Rightarrow u(x,y)=-xy$ в $O_\varepsilon(M)\Rightarrow$

$\Rightarrow \frac{\partial u}{\partial x}(0,y)=\frac{\partial}{\partial x}(-xy)\Big|_{x=0}=\underline{\underline{-y}}$

Пусть $y=0 \Rightarrow \frac{\partial u}{\partial x}(0,0)\equiv\frac{d}{dx}[\underbrace{u(x,0)}_{\in G_1}]_{x=0}=$

$=\frac{d}{dx}(+x\cdot 0)\Big|_{x=0}=0=\underline{\underline{-y}}$ (т.к. $y=0$)

Т.о. $\forall y \Rightarrow \frac{\partial u}{\partial x}(0,y)=-y$

Тогда $u_{xy}(0,0)=\frac{d}{dy}\left[\frac{\partial u}{\partial x}(0,y)\right]_{y=0}=$

$=\frac{d}{dy}(-y)\Big|_{y=0}=-1$ Ан-но $u_{yx}(0,0)=+1$

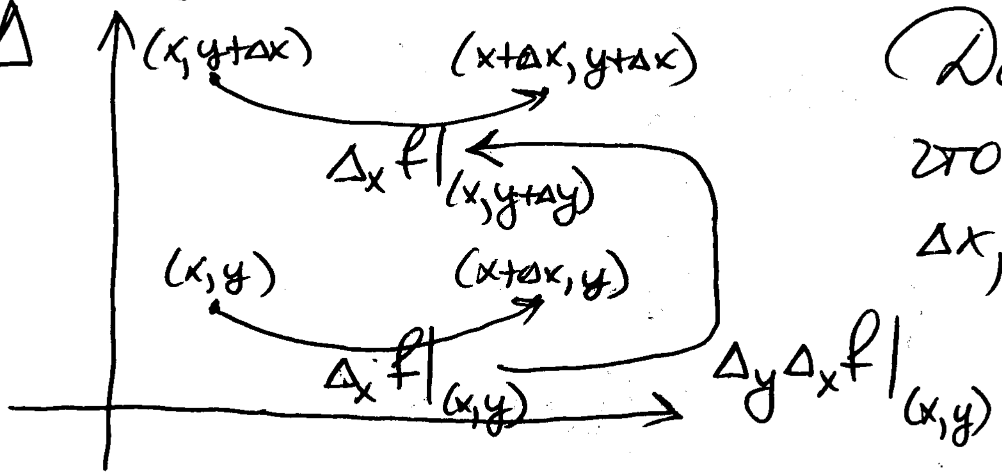
6.6

Опр Ф-я $f(x,y)$ наз дважды диф-й в т. M, если f_x и f_y диф в т. M

Теор 18 (о = смеш-х пр-ых)

Если $f(x,y)$ дважды диф в т. $M(x,y)$, то $f_{xy}(x,y)=f_{yx}(x,y)$

Δ



Далее считаем, что x, y — фикс, $\Delta x, \Delta y$ — перем

Рассм

$$\Delta_y \Delta_x f\Big|_{(x,y)} \equiv \Delta_x f\Big|_{(x,y+\Delta y)} - \Delta_x f\Big|_{(x,y)} =$$

$$= \left[f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)\right] - \left[f(x+\Delta x, y) - f(x,y)\right] =$$

$$= \left[f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y)\right] - \left[f(x, y+\Delta y) - f(x,y)\right] =$$

$$= \Delta_y f\Big|_{(x+\Delta x, y)} - \Delta_y f\Big|_{(x,y)} \equiv \Delta_x \Delta_y f\Big|_{(x,y)}$$

Пусть Δx — фикс, Δy — перем

$$\Rightarrow \Delta_y \Delta_x f\Big|_{(x,y)} = \underbrace{[f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)]}_{F(y+\Delta y)} -$$

$$- \underbrace{[f(x+\Delta x, y) - f(x,y)]}_{F(y)} \underset{\text{теор Л-жа}}{=} F'_y(y+\underset{\in(0,1)}{\theta}\Delta y)\Delta y =$$

$$= \Delta y\,[f_y(x+\Delta x, y+\theta\Delta y) - f_y(x, y+\theta\Delta y)] =$$

$$= \Delta y\,\{\underbrace{[f_y(x+\Delta x, y+\theta\Delta y) - f_y(x,y)]}_{\Delta f_y(\Delta x,\, \theta\Delta y)} -$$

$$- \underbrace{[f_y(x, y+\theta\Delta y) - f_y(x,y)]}_{\Delta f_y(0,\, \theta\Delta y)}\} =$$

$$= \Delta y\,\{\text{исп диф } f_y \text{ в т. } M(x,y)\} =$$

$$= \Delta y\,\{f_{yx}(x,y)\Delta x + \cancel{f_{yy}(x,y)\theta\Delta y} + \alpha_1\Delta x + \beta_1\theta\Delta y -$$

$$- \cancel{f_{yy}(x,y)\theta\Delta y} - \beta_2\theta\Delta y\} = f_{yx}(x,y)\Delta x\Delta y +$$

$$+ \alpha_1\Delta x\Delta y + \theta\beta_1\Delta y^2 - \theta\beta_2\Delta y^2$$

Положим $\Delta x = \Delta y = h$ $\quad \alpha(h)\to 0,\ h\to 0$

$$\Rightarrow \Delta_y\Delta_x f\Big|_{(x,y)} = f_{yx}(x,y)\,h^2 + \overbrace{(\alpha_1 + \underset{\text{огр}}{\theta}\beta_1 - \underset{\text{огр}}{\theta}\beta_2)}^{\alpha(h)}\,h^2 =$$

$$= [f_{yx}(x,y) + \alpha]\,h^2$$

Старый вариант страницы 6.7 → 6.7+

$$\Rightarrow \Delta_y \Delta_x f\Big|_{(x,y)} = \underbrace{\Delta_x f\Big|_{(x,y+\Delta y)}}_{F(y+\Delta y)} - \underbrace{\Delta_x f\Big|_{(x,y)}}_{F(y)} =$$

теор Л-жа

$$= F'_y(y+\underset{\in(0,1)}{\theta}\Delta y)\,\Delta y = \Delta y \frac{\partial}{\partial y}\Big[\Delta_x f\Big|_{(x,\,y+\theta\Delta y)}\Big] =$$

$$= \Delta y \frac{\partial}{\partial y}\big[f(x+\Delta x, y+\theta\Delta y) - f(x, y+\theta\Delta y)\big] =$$

$$-f_y(x,y) \qquad +f_y(x,y)$$

$$= \Delta y \Big\{ \underbrace{\big[f_y(x+\Delta x, y+\theta\Delta y) - f_y(x,y)\big]}_{\Delta f_y(\Delta x,\,\theta\Delta y)} - \underbrace{\big[f_y(x, y+\theta\Delta y) - f_y(x,y)\big]}_{\Delta f_y(0,\,\theta\Delta y)} \Big\} =$$

$$= \Delta y \{ \text{исп дифф } f_y \text{ в т. } M(x,y) \} =$$

$$= \Delta y \{ f_{yx}(x,y)\Delta x + \cancel{f_{yy}(x,y)\theta\Delta y} + \alpha_1\Delta x + \beta_1\theta\Delta y - \cancel{f_{yy}(x,y)\theta\Delta y} - \beta_2\theta\Delta y \} = f_{yx}(x,y)\Delta x\Delta y +$$

$$+ \alpha_1\Delta x\Delta y + \theta\beta_1\Delta y^2 - \theta\beta_2\Delta y^2$$

Положим $\Delta x = \Delta y = h$

$$\Rightarrow \Delta_y\Delta_x f\Big|_{(x,y)} = f_{yx}(x,y)h^2 + \overbrace{(\alpha_1 + \underset{\text{огр}}{\theta}\beta_1 - \underset{\text{огр}}{\theta}\beta_2)}^{\alpha(h)\to 0,\ h\to 0}h^2 =$$

$$= [f_{yx}(x,y) + \alpha]\,h^2$$

Старый вариант страницы 6.7

Ан-но $\Delta_x \Delta_y f\big|_{(x,y)} = [f_{xy}(x,y)+\beta]h^2$ 6.8

Т.к. $\Delta_y \Delta_x \square = \Delta_x \Delta_y \square$, то

$$f_{yx}(x,y)+\underbrace{\alpha}_{\to 0} = f_{xy}(x,y)+\underbrace{\beta}_{\to 0} \quad h\to 0$$

$\Rightarrow f_{yx}(x,y)=f_{xy}(x,y)$ ⊿

Диф-лы высших порядков

Пусть $u(x,y)$ дважды диф в т. $M_0(x_0,y_0)$

x, y – нез-ые пер-ые

Нап, что $du\big|_{M_0} \equiv \frac{\partial u}{\partial x}(M_0)dx + \frac{\partial u}{\partial y}(M_0)dy =$

$$= \underbrace{\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}\right)}_{\equiv d} u\Big|_{M_0}$$

Обозн

$$\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}\right)^2 \equiv dx^2\frac{\partial^2}{\partial x^2}+2dxdy\frac{\partial^2}{\partial x\partial y}+dy^2\frac{\partial^2}{\partial y^2}$$

Опр $d^2u\big|_{M_0} \equiv \left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}\right)^2 u\Big|_{M_0} =$

$= u_{xx}(M_0)dx^2+2u_{xy}(M_0)dxdy+u_{yy}(M_0)dy^2$

(Старшие диф-лы опр-ся ан-но)