Глава II. Некоторые классические задачи математической физики

§4. Динамика сорбции газа

Для проверки условия $u(0,t)=u_0$ (15) сделаем во втором слагаемом правой части формулы (20) замену $\theta=\dfrac{\tau}{x_1}$. Тогда получим:

$$u=u_0e^{-x_1}\left\{e^{-t_1}I_0(2\sqrt{x_1t_1})+\int_0^{t_1}e^{-\theta}I_0(2\sqrt{x_1\theta})d\theta\right\}$$

Положим теперь $x_1=0$:

$$u=u_0\left\{e^{-t_1}+\int_0^{t_1}e^{-\theta}d\theta\right\}=u_0\left\{e^{-t_1}-e^{-t_1}+1\right\}=u_0$$

Выполнение условия (17) очевидно:

$$u(x,0)=u_0e^{-x_1}=u_0e^{-\frac{\beta}{v}x}$$

$$t' = t - \frac{x}{v},\; x' = x;\; t'_0 = t_0 - \frac{x}{v} = 0 \;\Rightarrow\; t_0 = \frac{x}{v}$$

$$-v\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + \frac{\partial a}{\partial t} \;(7);\; \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'} + \frac{\partial u}{\partial t'}\frac{\partial t'}{\partial x} \;\Rightarrow$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'} - \frac{1}{v}\frac{\partial u}{\partial t'};\; (7) \;\Rightarrow\; -v\frac{\partial u}{\partial x'} + \cancel{\frac{\partial u}{\partial t'}} = \cancel{\frac{\partial u}{\partial t'}} + \frac{\partial a}{\partial t'} \;\Rightarrow\; -v\frac{\partial u}{\partial x'} = \frac{\partial a}{\partial t'} \;(22)$$

$$\frac{\partial a}{\partial t} = \beta(u - \gamma a) \;(8);\; \frac{\partial a}{\partial t} = \frac{\partial a}{\partial t'};\; (8) \;\Rightarrow\; \frac{\partial a}{\partial t'} = \beta(u - \gamma a) \;(23)$$

$$\xi = \frac{x'\beta}{v},\; \tau = t'\beta \;(24);\; \frac{\partial u}{\partial x'} = \frac{\partial u}{\partial \xi}\frac{\beta}{v};\; \frac{\partial a}{\partial t'} = \frac{\partial a}{\partial \tau}\beta$$

$$(22) \;\Rightarrow\; -v\frac{\partial u}{\partial \xi}\frac{\beta}{v} = \frac{\partial a}{\partial \tau}\beta \;\Rightarrow\; -\frac{\partial u}{\partial \xi} = \frac{\partial a}{\partial \tau} \;\Rightarrow\; \frac{\partial u}{\partial \xi} + \frac{\partial a}{\partial \tau} = 0 \;(25)$$

$$(23) \;\Rightarrow\; \frac{\partial a}{\partial \tau}\beta = \beta(u - \gamma a) \;\Rightarrow\; \frac{\partial a}{\partial \tau} = u - \gamma a \;(26)$$

$$t = t' + \frac{x'}{v} = \frac{1}{\beta}\tau + \frac{1}{\beta}\xi = 0 \;\Rightarrow\; \tau + \xi = 0, \tau = -\xi$$

$$a(x,0) = 0 \;(9) \;\Rightarrow\; a\big|_{\tau=-\xi} = 0 \;(27)$$

$$u(x,0) = 0 \;(10) \;\Rightarrow\; u\big|_{\tau=-\xi} = 0 \;(28)$$

$$u(0,t) = u_0 \;(11) \;\Rightarrow\; u\big|_{\xi=0} = u_0 \;(29)$$

$$t' = t - \frac{x}{v}, \quad x' = x; \quad \xi = \frac{x'\beta}{v}, \quad \tau = t'\beta,$$

$$(25) \Rightarrow \frac{\partial^2 u}{\partial\xi\partial\tau} = -\frac{\partial^2 a}{\partial\tau^2}; \quad (26) \Rightarrow \frac{\partial^2 a}{\partial\tau^2} = \frac{\partial u}{\partial\tau} - \gamma\frac{\partial a}{\partial\tau}$$

$$\frac{\partial^2 u}{\partial\xi\partial\tau} = -\frac{\partial u}{\partial\tau} + \gamma\frac{\partial a}{\partial\tau}; \quad (25) \Rightarrow \frac{\partial a}{\partial\tau} = -\frac{\partial u}{\partial\xi} \Rightarrow \frac{\partial^2 u}{\partial\xi\partial\tau} + \frac{\partial u}{\partial\tau} + \gamma\frac{\partial u}{\partial\xi} = 0 \quad (30)$$

$$(7)-(10) \Rightarrow (10) \Rightarrow u(x,0) = 0; \; (8) \Rightarrow \frac{\partial a}{\partial t}(x,0) = \beta(\underbrace{u(x,0)}_{=0\ (10)} - \gamma \underbrace{a(x,0)}_{=0\ (9)}) = 0$$

$$(10) \Rightarrow \frac{\partial u}{\partial x}(x,0) = 0; \; (7) \Rightarrow \frac{\partial u}{\partial t}(x,0) = -v\underbrace{\frac{\partial u}{\partial x}(x,0)}_{=0\ (10)} - \underbrace{\frac{\partial a}{\partial t}(x,0)}_{=0\ (8)} = 0;$$

$$u(x,0) = 0; \quad \frac{\partial u}{\partial t}(x,0) = 0; \quad x = \frac{v}{\beta}\xi; \quad t = t' + \frac{x}{v} = \frac{1}{\beta}\tau + \frac{1}{\beta}\xi;$$

$$\frac{\partial u}{\partial\xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial\xi} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial\xi} = \frac{v}{\beta}\frac{\partial u}{\partial x} + \frac{1}{\beta}\frac{\partial u}{\partial t} \Rightarrow \frac{\partial u}{\partial\xi}\Big|_{\tau=-\xi} = \frac{v}{\beta}\underbrace{\frac{\partial u}{\partial x}(x,0)}_{=0} + \frac{1}{\beta}\underbrace{\frac{\partial u}{\partial t}(x,0)}_{=0} = 0$$

$$\frac{\partial u}{\partial\tau} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial\tau} = \frac{1}{\beta}\frac{\partial u}{\partial t} \Rightarrow \frac{\partial u}{\partial\tau}\Big|_{\tau=-\xi} = \frac{1}{\beta}\underbrace{\frac{\partial u}{\partial t}(x,0)}_{=0} = 0$$

$$\frac{\partial u}{\partial n}\Big|_{\tau=-\xi} = \left(\frac{\partial u}{\partial\tau}\cos(n\hat{,}\tau) + \frac{\partial u}{\partial\xi}\cos(n\hat{,}\xi)\right)_{\tau=-\xi} = 0$$

$$u\Big|_{\tau=-\xi} = u(x,0) = 0$$

$$\begin{cases} \dfrac{\partial^2 u}{\partial\xi\partial\tau} + \dfrac{\partial u}{\partial\tau} + \gamma\dfrac{\partial u}{\partial\xi} = 0, \quad (\xi,\tau) \in D \quad (30) \\ u\Big|_{\tau=-\xi} = 0 \quad \Rightarrow \quad u = 0, \quad (\xi,\tau) \in D \\ \dfrac{\partial u}{\partial n}\Big|_{\tau=-\xi} = 0, \quad (\xi,\tau) \in D \end{cases}$$

$$\frac{\partial a}{\partial t} = -v\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0 \Rightarrow a = \frac{\beta}{\gamma}u - \frac{1}{\gamma}\frac{\partial a}{\partial t} = 0$$

$$\xi=\frac{\beta}{v}x,\ \theta=\beta\gamma t\ \ (35)\ \Rightarrow\ \frac{\partial^2 u}{\partial x\partial t}=\frac{\beta}{v}\beta\gamma\frac{\partial^2 u}{\partial\xi\partial\theta};\ \frac{\beta}{v}\frac{\partial u}{\partial t}=\frac{\beta}{v}\beta\gamma\frac{\partial u}{\partial\theta};\ \beta\gamma\frac{\partial u}{\partial x}=\beta\gamma\frac{\beta}{v}\frac{\partial u}{\partial\xi};$$

$$u_{xt}+\frac{\beta}{v}u_t+\beta\gamma u_x=0\ \ (16)\ \Rightarrow\ \frac{\beta^2}{v}\gamma u_{\xi\theta}+\frac{\beta}{v}\beta\gamma u_\theta+\beta\gamma\frac{\beta}{v}u_\xi=0\ \Rightarrow\ u_{\xi\theta}+u_\theta+u_\xi=0\ \ (36)$$

$$u(\xi,\theta)=w(\xi,\theta)e^{-\xi-\theta}\quad(39)\ \Rightarrow$$

$$\frac{\partial u}{\partial\theta}=(\frac{\partial w}{\partial\theta}-w)e^{-\xi-\theta}\ ;\quad \frac{\partial u}{\partial\xi}=(\frac{\partial w}{\partial\xi}-w)e^{-\xi-\theta}$$

$$\frac{\partial^2 u}{\partial\xi\partial\theta}=(\frac{\partial^2 w}{\partial\xi\partial\theta}-\frac{\partial w}{\partial\theta}-\frac{\partial w}{\partial\xi}+w)e^{-\xi-\theta}\quad\Rightarrow$$

$$w_{\xi\theta}-w=0\qquad(40)$$

$$u(x,0)=u_0e^{-\frac{\beta}{v}x}\ \Rightarrow\ u(\xi,0)=e^{-\xi}\ \ (37)$$

$$при\ \ u_0=1:\quad u(0,\theta)=1\ \ (38);\quad (37),(38)\ \Rightarrow$$

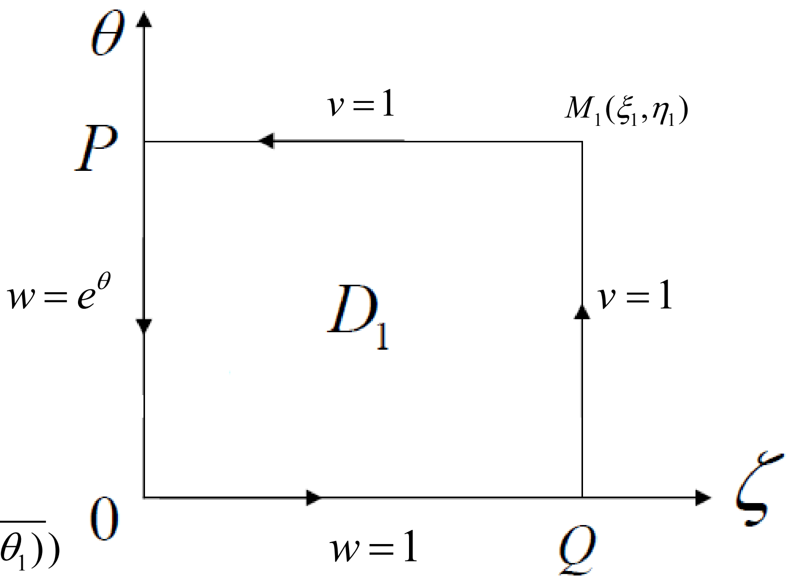
$$u(\xi,0)=w(\xi,0)e^{-\xi}=e^{-\xi}\ \Rightarrow\ w(\xi,0)=1\ \ (41)$$

$$u(0,\theta)=w(0,\theta)e^{-\theta}=1\ \Rightarrow\ w(0,\theta)=e^{\theta}\ \ (42)$$

Функция Римана для уравнения (40):

$$v(\xi,\theta,\xi_1,\theta_1)=I_0(2\sqrt{(\xi-\xi_1)(\theta-\theta_1)})\quad(43)$$

$$\begin{cases} w_{\xi\theta} - w = 0 \\ w(\xi, 0) = 1 \\ w(0,\theta) = e^{\theta} \end{cases}$$



$$v(\xi,\theta,\xi_1,\theta_1) = I_0(2\sqrt{(\xi-\xi_1)(\theta-\theta_1)})$$

$$\begin{cases} v_{\xi\theta} - v = 0 \quad в \ D_1 \ (45) \\ v = 1 \ \ на \ PM_1 \ и \ M_1Q \ (46) \end{cases}$$

$$\int\limits_{D_1} (w\overset{=v}{v_{\xi\theta}} - \overset{=w}{w_{\xi\theta}}v)d\xi d\theta = \int\limits_{D_1} (wv - wv)d\xi d\theta = 0 \ \Rightarrow$$

$$\underbrace{\int\limits_0^Q (\overset{=1}{v_\xi w} - \overset{=0}{vw_\xi})d\xi}_{\text{I}} + \underbrace{\int\limits_Q^{M_1} (\overset{=1}{vw_\theta} - \overset{=0}{v_\theta w})d\theta}_{\text{II}} + \underbrace{\int\limits_{M_1}^P (\overset{=0}{v_\xi w} - \overset{=1}{vw_\xi})d\xi}_{\text{III}} + \underbrace{\int\limits_P^0 (\overset{=e^\theta}{vw_\theta} - \overset{=e^\theta}{v_\theta w})d\theta}_{\text{IV}} = 0$$

I) $$\int\limits_0^Q \overset{=1}{v_\xi w}d\varsigma = \int\limits_0^Q v_\xi d\xi = v\Big|_0^Q = v(Q) - v(0) = I_0(0) - I_0(2\sqrt{\xi_1\theta_1}) = 1 - I_0(2\sqrt{\xi_1\theta_1}) \quad (49)$$

($\xi = \xi_1$)

II) $$\int\limits_Q^M \overset{=1}{v}w_\theta d\theta = \int\limits_Q^{M_1} w_\theta d\theta = v\Big|_0^Q = w(M_1) - w(Q) = w(M_1) - 1 \quad (50)$$

III) $$-\int\limits_{M_1} \overset{=1}{v}w_\xi d\xi = -\int\limits_{M_1}^P w_\xi d\xi = w(M_1) - w(P) = w(M_1) - e^{\theta_1} \quad (51)$$

$$\text{IV)}\ \int_P^0 e^{\theta}(v-v_{\theta})d\theta=\int_P^0 e^{\theta}vd\theta-\int_P^0 e^{\theta}v_{\theta}d\theta=\int_P^0 e^{\theta}vd\theta-e^{\theta}v\Big|_P^0+\int_P^0 e^{\theta}vd\theta=$$

$$=(e^{\theta}v)(P)-(e^{\theta}v)(0)+2\int_P^0 e^{\theta}vd\theta=e^{\theta_1}-I_0(2\sqrt{\xi_1\theta_1})+2\int_P^0 e^{\theta}I_0(2\sqrt{(\xi-\xi_1)(\theta-\theta_1)})d\theta \quad (52)$$

$$I-IV \quad \Rightarrow \quad 0=\cancel{1}-I_0(2\sqrt{\xi_1\theta_1})+w(M_1)-\cancel{1}+w(M_1)-\cancel{e^{\theta_1}}+\cancel{e^{\theta_1}}-I_0(2\sqrt{\xi_1\theta_1})+$$

$$+2\int_P^0 e^{\theta}I_0(2\sqrt{(\xi-\xi_1)(\theta-\theta_1)})d\theta \quad \Rightarrow$$

$$w(M_1)=I_0(2\sqrt{\xi_1\theta_1})-\int_{\theta_1}^0 e^{\theta}I_0(2\sqrt{(0-\xi_1)(\theta-\theta_1)})d\theta \quad \Rightarrow$$

$$w(M_1)=I_0(2\sqrt{\xi_1\theta_1})-\int_{\theta_1}^0 e^{\theta}I_0(2\sqrt{\xi_1(\theta_1-\theta)})d\theta=I_0(2\sqrt{\xi_1\theta_1})+\int_0^{\theta_1} e^{\theta}I_0(2\sqrt{\xi_1(\theta_1-\theta)})d\theta \quad (53)$$

$$(39) \Rightarrow u(M_1)=u(\xi_1,\theta_1)=e^{-\xi_1-\theta_1}w(M_1) \quad (54)$$

$$(53),(54) \Rightarrow u(\xi_1,\theta_1)=e^{-\xi_1}\left\{e^{-\theta_1}I_0(2\sqrt{\xi_1\theta_1})+\int_0^{\theta_1} e^{\theta-\theta_1}I_0(2\sqrt{\xi_1(\theta_1-\theta)})d\theta\right\} \quad (55)$$

$$\text{Замена}: \quad \tau = \xi_1(\theta_1 - \theta) \Rightarrow \theta_1 - \theta = \frac{\tau}{\xi_1} \Rightarrow d\theta = -\frac{1}{\xi_1} d\tau \Rightarrow$$

$$\int_0^{\theta_1} e^{\theta - \theta_1} I_0(2\sqrt{\xi_1(\theta_1 - \theta)})d\theta = -\frac{1}{\xi_1}\int_{\xi_1\theta_1}^{0} e^{-\frac{\tau}{\xi_1}} I_0(2\sqrt{\tau})d\tau = \frac{1}{\xi_1}\int_0^{\xi_1\theta_1} e^{-\frac{\tau}{\xi_1}} I_0(2\sqrt{\tau})d\tau \Rightarrow$$

$$u(\xi_1, \theta_1) = e^{-\xi_1}\left\{ e^{-\theta_1} I_0(2\sqrt{\xi_1\theta_1}) + \frac{1}{\xi_1}\int_0^{\xi_1\theta_1} e^{-\frac{\tau}{\xi_1}} I_0(2\sqrt{\tau})d\tau \right\} \qquad (56)$$

В формуле (20): $\xi_1 = x_1, \theta_1 = t_1$ *поэтому формула* (56) *совпадает с формулой* (20) *при* $u_0 = 1$